Regularized Casimir energy for an infinite dielectric cylinder subject to light-velocity conservation

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Abstract. The Casimir energy of a dilute dielectric cylinder, with the same light-velocity as in its surrounding medium, is evaluated exactly to first order in $\xi^2 = \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right)^2$ (where $\varepsilon_1, \varepsilon_2$ are the dielectric constants of the cylinder and of its environment), and numerically to higher orders in ξ^2. The first part is carried out using addition formulas for Bessel functions, and no Debye expansions are required.

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1 Introduction

Zero-point fluctuations of quantum fields give rise to forces, which are regarded as manifestations of the Casimir effect (for reviews, see e.g. refs.[1]). From the theoretical viewpoint, one of the most daunting aspects of the evaluation of Casimir energies, even for highly symmetrical boundaries, is its sheer difficulty. Many mathematical methods has been developed, but even the simplest ones demand considerable efforts.

At the core of several of these techniques one finds uniform asymptotic expansions —also called *Debye* expansions— of Bessel or Ricatti-Bessel functions appearing in integrals over momentum-like variables. This fruitful method was used as early as —at least— the time of ref.[2], and has been repeatedly revisited in a huge number of articles, often in the framework of other regularization schemes (see e.g. ref.[3] and refs. therein). However reliable, the whole Debye expansion technique is a time-consuming process and the search for computational alternatives might be of interest [4]. This is, precisely, one of the motivations of the present letter. Our purpose is to take further the exploitation of summation theorems for Bessel functions, started in ref. [4] for cases with spherical surfaces, and apply it to a

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problem with a cylindrical boundary.

We are considering a material cylinder of radius a, infinitely long, placed along the z-axis, with permitivitty and permeability ε_1, μ_1, surrounded by a medium with permitivitty and permeability ε_2, μ_2. For such surfaces, a special situation is the case where the light velocities in both media —interior (1) and exterior (2)— are the same, i.e.,

$$\varepsilon_1 \mu_1 = \varepsilon_2 \mu_2 \equiv c^{-2}, \tag{1}$$

where c is the common light-velocity. Since any variation in ε affects μ, this is called *dielectric-diamagnetic* case, as opposed to the *purely dielectric* one, in which $\mu_1 = \mu_2 = 1$ but the velocity has to change. Dielectric-diamagnetic conditions are often desirable as they cause the frequency equations to simplify and some divergences to cancel out. In a QCD context, ε and μ refer to colour permitivitty and permeability (see [5] and refs. therein). Illustrations of the dependence of the interquark potential on the boundary conditions for a string model have been provided in ref.[6].

In refs. [7], [8] and [9] the regularized Casimir energy per lateral unit-length for an infinite dielectric-diamagnetic cylinder has been studied. Up to the order of

$$\xi^2 = \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \right)^2, \tag{2}$$

the energy has been shown to vanish within all the tested degrees of numerical accuracy. The next contribution, which is of the order of ξ^4, has been found —to our knowledge, for the first time— in ref.[9].

In our minds, medium 2 will be pure vacuum and medium 1 a very tenuous dielectric, which means $\varepsilon_2 = \mu_2 = 1$ and $\varepsilon_1 - 1 \ll 1$. As a result, the ξ^2 parameter, defined by eq.(2), is a small number. According to ref.[7], the eigenfrequencies ω coming from the Maxwell equations for this problem are given by the zeros of some equations of the type $f_n(k_z, \omega, a) = 0$, $n \in \mathbf{Z}$ —eqs.(2.3)-(2.5) in ref.[7]—. Further, in cases where the relation (1) holds, f_n takes the form

$$f_n(k_z, \omega, a) = -a^2 c^{-2} \lambda^6 \frac{(\varepsilon_1 + \varepsilon_2)^2}{4\, \varepsilon_1\, \varepsilon_2} \left[\xi^2 \mathcal{P}_n^2(\lambda a) + \frac{4}{\pi^2 (\lambda a)^2} \right], \quad \mathcal{P}_n(x) \equiv \left(J_n H_n \right)'(x), \tag{3}$$

where ξ^2 is given by (2), J_n, H_n are Bessel and Hankel functions, and

$$\omega = c \sqrt{\lambda^2 + k_z^2}. \tag{4}$$

Every λ belongs to the eigenfrequency set of the projected two-dimensional problem —say Λ—, while $-\infty < k_z < \infty$, i.e., the values of k_z are continuous without any restriction. Before regularizing, the Casimir energy per unit-length ($\mathcal{E}_C$) is given by the mode sum

$$\mathcal{E}_C = \frac{1}{2}\hbar \sum_{n,m} \int \frac{dk_z}{2\pi}\, \omega_{n,m,k_z}, \qquad\qquad \omega_{n,m,k_z} = c\sqrt{\lambda_{n,m}^2 + k_z^2}. \tag{5}$$

The n-index is the angular momentum number, while m describes the remaining degree of freedom, i.e., labels the different λ-values at a given n.

The present work is organized as follows. In sec. 2 we follow ref. [7] and evaluate the energy density $g^{(2)}$ (in momentum space) up to the order of ξ^2, by a modified Bessel function summation theorem, and resorting to the properties of Meijer G functions. Then, we show that the integration of $g^{(2)}$ yields a vanishing result. Sec. 3 is devoted to an alternative approach based on a zeta function prescription for the initial mode sum like in refs.[3, 10]. Apart from proving to be easier, this technique paves the way to the numerical calculation of higher order contributions. Our conclusions are given in sec. 4.

2 Density method

We begin by reviewing the procedure used in ref. [7] and obtaining an expression for the Casimir energy. The mode sum is first represented, as usual in these cases, by a contour integral

$$E_C = -\frac{\hbar c}{2}\int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i}\frac{1}{2}\oint_C \sqrt{\lambda^2 + k_z^2}\, d_\lambda \ln\left[\frac{f_n(k_z,\omega,a)}{f_{n,as}(k_z,\omega)}\right], \tag{6}$$

where the integration contour C consists of a straight line parallel to, and just to the right of, the imaginary axis, $(-i\infty, +i\infty)$ closed by a semicircle of an infinitely large radius in the right half-plane. The branch line of the function $\varphi(\lambda) = \sqrt{\lambda^2 + k_z^2}$ is chosen to run between $-i|k_z|$ and $i|k_z|$ on the imaginary axis. In terms of $y = \operatorname{Im}\lambda$ we have

$$\varphi(iy) = \begin{cases} i\sqrt{y^2 - k_z^2}, & y > k_z, \\ \pm\sqrt{k_z^2 - y^2}, & |y| < k_z, \\ -i\sqrt{y^2 - k_z^2}, & y < -k_z. \end{cases} \tag{7}$$

Noting that the argument of the logarithm is an even function of iy, (6) reduces to

$$E_C = -\frac{\hbar c}{2\pi^2}\sum_{n=-\infty}^{\infty}\int_0^{\infty} dk_z \int_{k_z}^{\infty} \sqrt{y^2 - k_z^2}\, d_y \ln\left[1 - \xi^2 (y\partial_y (I_n(ay)K_n(ay)))^2\right], \tag{8}$$

where we have expressed (3) explicitly on the imaginary axis. Integrating with respect to k_z we obtain the Casimir energy per unit length ($\mathcal{E}_\mathcal{C}$) as an integral over y, namely

$$\mathcal{E}_\mathcal{C} = \frac{\hbar c}{4\pi} \sum_{n=-\infty}^{\infty} \int_0^\infty dy\, y^2 \partial_y \ln \left[1 - \xi^2 (y\partial_y (I_n(ay) K_n(ay)))^2\right] . \tag{9}$$

In the following subsections, we use the approach of ref. [4] to evaluate this expression.

2.1 Density calculation to the order of ξ^2

Having integrated k_z out, the integrand in expression (9) may be interpreted as the density of Casimir energy with respect to the parameter $y = i\lambda$. This density may be evaluated by expanding in terms of ξ^2, the first contribution being

$$g^{(2)}(y) \equiv \frac{\hbar c \xi^2}{4\pi} \sum_{n=-\infty}^{\infty} dy\, y^2 \partial_y [y \partial_y (I_n(ay) K_n(ay))]^2. \tag{10}$$

We shall now show that $g^{(2)}$ can be calculated explicitly by a variant of the method shown in ref. [4]. Specifying the identity 8.530.2 of ref. [11] to Hankel solutions $Z_n = H_n^{(1)} \equiv H_n$, and choosing the ν parameter equal to zero, we obtain the summation theorem

$$H_0(m\, R(\rho, r, \varphi)) = \sum_{n=-\infty}^{\infty} J_n(m\rho)\, H_n(mr)\, e^{in\varphi}, \qquad R(\rho, r, \varphi) \equiv \sqrt{\rho^2 + r^2 - 2\rho r \cos\varphi}. \tag{11}$$

Performing the change $m \to im$, and selecting the special case $\rho = r$, it becomes

$$K_0(m\, R(r, \varphi)) = \sum_{n=-\infty}^{\infty} I_n(mr)\, K_n(mr)\, e^{in\varphi}, \qquad R(r, \varphi) \equiv r\sqrt{2(1 - \cos\varphi)} = 2r\, |\sin(\varphi/2)| . \tag{12}$$

Differentiating with respect to m, using the property $K_0'(z) = -K_1(z)$ together with the fact that $K_n = K_{-n}$, and setting $m = 1$ afterwards, we have

$$-R(r, \varphi)\, K_1(R(r, \varphi)) = \sum_{n=-\infty}^{\infty} r\, (I_n\, K_n)'(r)\, e^{in\varphi}. \tag{13}$$

Recalling the orthogonality of the imaginary exponential functions, we arrive at

$$\frac{1}{2\pi} \int_0^{2\pi} d\varphi\, \left[R(r, \varphi)\, K_1(R(r, \varphi))\right]^2 = \sum_{n=-\infty}^{\infty} \left[r (I_n\, K_n)'(r)\right]^2 \equiv F(r), \tag{14}$$

In order to proceed, we rename r into ay, and do the variable change $u \equiv |\sin(\varphi/2)|$. After differentiation with respect to y, we realize that the sum in (10) is given by

$$g^{(2)}(y) = \frac{\hbar c \xi^2}{4\pi} \sum_{n=-\infty}^{\infty} y^2 \partial_y \left[y \partial_y (I_n(ay) K_n(ay))\right]^2 = \frac{2\hbar c \xi^2}{\pi^2} \int_0^1 du\, \frac{u^2}{\sqrt{1 - u^2}} y^2 \partial_y (ay K_1(2ayu))^2, \tag{15}$$

Thus, we have turned the problem of calculating an infinite angular-momentum summation into the evaluation of a definite integral of a transcendental function. Next, by writing the product of Bessel functions appearing in Eq.(15) in terms of the Meijer G function [12], $g^{(2)}$ can be evaluated explicitly. (See Appendix for the definition and some simple properties of this function.) First, we write

$$u^2a^2y^2K_1^2(2ayu) = u^2a^2y^2\frac{\sqrt{\pi}}{2}G_{13}^{30}\left(\frac{1}{2};-1,0,1;4a^2y^2u^2\right) = \frac{\sqrt{\pi}}{8}G_{13}^{30}\left(\frac{3}{2};0,1,2;4a^2y^2u^2\right) \tag{16}$$

where, in the last step, we have made use of the identity (48). Differentiating (16) with respect to y and substituting in (15) we obtain

$$g^{(2)}(y) = -\frac{2\hbar c\xi^2}{\pi^{3/2}}\int_0^1 du\,\frac{y^3a^2u^2}{\sqrt{1-u^2}}G_{13}^{30}\left(\frac{1}{2};0,0,1;4a^2y^2u^2\right). \tag{17}$$

Using eq.5.5.2(5) in [12], and some straightforward manipulation one gets:

$$g^{(2)}(y) = -\frac{\hbar c\xi^2}{8\pi a}\,G_{24}^{31}\left(1,2;\frac{3}{2},\frac{3}{2},\frac{5}{2},\frac{1}{2};4a^2y^2\right). \tag{18}$$

2.2 Calculation of the energy to order ξ^2.

We now turn to the question of deriving the Casimir energy. There are two possibilities:

1. To perform the u integration in (15) after integrating over the y variable. Unfortunately, this turns out to be divergent. Considering the integral

$$\frac{2}{\pi^2}\int_0^\infty y^2dy\int_0^{\frac{\pi}{2}} du\,\partial_y[y\sin uK_1(2y\sin u)]^2 \tag{19}$$

and interchanging the order of integration, one arrives at

$$\int_0^\infty dy\,y^2\partial_y\left[y\sin u\,K_1(2y\sin u)\right]^2 = \frac{-1}{12\,\sin^2 u}. \tag{20}$$

If we now try to do the u-integration, the integral diverges. This shows that some further regularization is called for. In fact, in sec. 3 we will go through the same sort of calculation, but with the advantage of having applied zeta function regularization from the outset.

2. Direct integration of $g^{(2)}(y)$. We use the following identity from ref. [12] (Vol 1, page 215).

$$\begin{aligned}&\int_0^\infty dy\,y^{-a}K_\nu(2\sqrt{y})\,G_{pq}^{mn}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;xy\right)\\&=\frac{1}{2}G_{p+2,q}^{m,n+2}\left(a-\frac{\nu}{2},a+\frac{\nu}{2},a_1,\ldots,a_p;b_1,\ldots,b_q;x\right)\end{aligned} \tag{21}$$

In order to take advantage of this formula, we note that $K_{\frac{1}{2}}(y) = \sqrt{\frac{2}{\pi y}}\, e^{-y}$. Changing to a variable $t = \frac{{\sigma_r}^2 y^2}{4}$ and inserting $K_{\frac{1}{2}}$, we can cast the energy per unit-length into the form

$$\mathcal{E}_C^{(2)}\,\xi^2 = \int_0^\infty g^{(2)}(y)dy = -\lim_{\sigma_r\to 0}\frac{\hbar c\xi^2}{2a^2\pi\sqrt{\pi}\sigma_r}\int_0^\infty dt\, t^{-\frac{1}{4}}K_{\frac{1}{2}}\left(2\sqrt{t}\right)G^{31}_{24}\left(1,2;\frac{3}{2},\frac{3}{2},\frac{5}{2},\frac{1}{2};\frac{16t^2}{{\sigma_r}^2}\right). \tag{22}$$

Although we have inserted $K_{\frac{1}{2}}$, as a mere technicality to help calculate the energy, one can think of using it as an *exponential* regulator[3] (see ref.[4]). However the convergence of the integral shows that the density we have derived is already regularized in some sense. One can now use (21) to get:

$$\mathcal{E_C}^{(2)} = -\frac{\hbar c}{4a^2\pi\sqrt{\pi}\sigma_r}G^{33}_{44}\left(0,\frac{1}{2},1,2;\frac{3}{2},\frac{3}{2},\frac{5}{2},\frac{1}{2};\frac{16}{{\sigma_r}^2}\right). \tag{23}$$

In order to check the asymptotics as $\sigma_r\to 0$, we use the property (49), together with the asymptotics (again from ref.[12]) $G(x) = \mathcal{O}(|x|^\beta)$ as $x\to 0$, for $p\le q$, and $\beta = \max\{\mathrm{Re}\, b_h\}$ for $h = 1,\ldots,m$. In our case we simply have

$$\mathcal{E_C}^{(2)} \propto \lim_{\sigma_r\to 0}\frac{1}{\sigma_r}\mathcal{O}(|\sigma_r^2|) = \lim_{\sigma_r\to 0}\mathcal{O}(\sigma_r) = 0. \tag{24}$$

Thus, the ξ^2-term is shown to vanish, confirming the conclusions of refs. [7], [8] and [9], without recourse to numerical evaluations.

3 Complete zeta function regularization

In this section we will take a different approach, based on the application of the complete zeta function method (see e.g. refs.[3, 10]) to the initial mode sum (5). The use of zeta functions for regularizing such sort of sums dates from the time of refs.[13]. In the version we shall now apply, the regularized value of the Casimir energy per unit-length is

$$\mathcal{E}_C = \lim_{s\to -1}\frac{1}{2}\hbar c\,\zeta_{\Omega(D=3)}(s) \tag{25}$$

where the zeta function $\zeta_{\Omega(D=3)}$ for the whole set of ω-modes in the three-dimensional problem —say Ω— is given by

$$\zeta_{\Omega(D=3)}(s) = \sum_{n,m}\int_{-\infty}^\infty\frac{dk_z}{2\pi}\left(\frac{\omega_{n,m,k_z}}{c}\right)^{-s}. \tag{26}$$

[3]Actually, it is not difficult to show that applying an exponential regulator in the form $e^{-\sigma_r\omega}$ to (6), before carrying the k_z integration, yields the same result as the one we derive.

First, one assumes that s is large enough for this function to make sense, with the final aim of setting $s=-1$ at the end (usually, one introduces in (25) an arbitrary mass scale, but, in this problem, it turns out to be unnecessary). Taking into account (5), we write

$$\zeta_{\Omega(D=3)}(s) = \sum_{n,m} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \left[\lambda_{n,m}^2 + k_z^2\right]^{-s/2} = \frac{1}{2\pi} \mathrm{B}\left(\frac{s-1}{2}, \frac{1}{2}\right) \sum_{n,m} \lambda_{n,m}^{-(s-1)}. \tag{27}$$

Let's consider the zeta function for the projected two-dimensional problem,i.e., for the Λ eigenmode set:

$$\zeta_{\Lambda(D=2)}(\sigma) = \sum_{n,m} \lambda_{n,m}^{-\sigma} = \sum_{n=0}^{\infty} d_n \zeta_n(\sigma), \qquad \begin{cases} d_0 = 1, \\ d_n = 2, \quad \text{for } n \geq 1, \end{cases} \tag{28}$$

where $\zeta_n(\sigma)$ stands for the nth *partial-wave* zeta function

$$\zeta_n(\sigma) = \sum_{m=1}^{\infty} \lambda_{n,m}^{-\sigma}. \tag{29}$$

Bearing this in mind, we put (27) as

$$\begin{aligned} \zeta_{\Omega(D=3)}(s) &= \frac{1}{2\pi} \mathrm{B}\left(\frac{s-1}{2}, \frac{1}{2}\right) \zeta_{\Lambda(D=2)}(s-1) \\ &= \frac{1}{2\pi} \left[\frac{\zeta_{\Lambda(D=2)}(-2)}{s+1} + \left(\ln(2) - \frac{1}{2}\right) \zeta_{\Lambda(D=2)}(-2) + \zeta'_{\Lambda(D=2)}(-2) + \mathcal{O}(s+1)\right], \end{aligned} \tag{30}$$

where an expansion around $s=-1$ has taken place. This was the method applied in ref.[10].

Eqs. (28), (29) hold for Re $\sigma > 1$, but they will have to be analytically continued to the neighbourhood of $\sigma = -2$ (note that $\sigma = s-1$). Such an analytic continuation is carried out by the contour integration method of refs.[14, 3]. To begin with, one takes

$$a^{-\sigma}\, \zeta_n(\sigma) = \frac{\sigma}{2\pi i} \int_C du\, u^{-\sigma-1} \ln\left[f_n(u)\right], \text{ for Re } \sigma > 1 \;, \tag{31}$$

where $f_n(u) \equiv f_n(k_z, \omega, a)$ with $u \equiv \lambda a$, and C is a circuit in the complex u-plane enclosing all the positive zeros of $f_n(u)$. In the desired limit this contour will be semicircular, with the straight parts along the imaginary axis, and adequately avoiding the origin. The first step (see e.g.[14]) is to examine the asymptotic behaviour $f_{n,\mathrm{as}}(u)$ of $f_n(u)$ for $|u| \to \infty$. If $f_{n,\mathrm{as}}(u)$ has no roots inside of C, we leave the eq.(31) unchanged by setting

$$a^{-\sigma}\, \zeta_n(\sigma) = \frac{\sigma}{2\pi i} \int_C du\, u^{-\sigma-1} \ln\left[\frac{f_n(u)}{f_{n,\mathrm{as}}(u)}\right]. \tag{32}$$

Going back to (3), we observe that, for large x, $\mathcal{P}_n^2(x) = \mathcal{O}\left(x^{-4}\right)$. Then, one can write

$$f_n(u) = f_{n,\mathrm{as}}(u)\,\left[1+\xi^2\frac{\pi^2 u^2}{4}\mathcal{P}_n^2(u)\right], \qquad f_{n,\mathrm{as}}(u) = -\frac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\,\varepsilon_2}\,\frac{c^{-2}}{\pi^2 a^4}\,u^4. \tag{33}$$

Therefore, eq.(32) translates into

$$a^{-\sigma}\zeta_n(\sigma) = \frac{\sigma}{2\pi i}\int_C du\, u^{-\sigma-1}\,\ln\left[1+\xi^2\frac{\pi^2}{4}\,u^2\mathcal{P}_n^2(u)\right]. \tag{34}$$

After realizing that only the vertical parts of C —where $u = e^{\pm i\pi/2}y$— are actually contributing to the integration, eq.(34) yields

$$\zeta_n(\sigma) = a^{\sigma}\,\frac{\sigma}{\pi}\sin\left(\frac{\pi\sigma}{2}\right)\int_0^\infty dy\, y^{-\sigma-1}\,\ln\left\{1-\xi^2\left[y\,\left(I_n\,K_n\right)'(y)\right]^2\right\}, \text{ for } -1<\mathrm{Re}\,\sigma<0, \tag{35}$$

where we have used $\mathcal{P}_n^2(\pm iy) = \dfrac{4}{\pi^2}\left[\left(I_n\,K_n\right)'(y)\right]^2$, being I_n, K_n the corresponding modified Bessel functions (note that this y is dimensionless). All this has validity near $\sigma=-1$, but we still need some further work in order to reach the neighbourhood of $\sigma=-2$.

3.1 Calculation to the order of ξ^2

Let $\mathcal{E}_C = \sum_{p\geq 1}\mathcal{E}_C^{(2p)}\,\xi^{2p}$, and analogously for the involved zeta functions. Then,

$$\begin{aligned}
\zeta_n(\sigma) &= -a^{\sigma}\,\frac{\sigma}{\pi}\sin\left(\frac{\pi\sigma}{2}\right)\sum_{p\geq 1}\frac{1}{p}\xi^{2p}\,A_n^{(2p)}(\sigma),\\
A_n^{(2p)}(\sigma) &= \int_0^\infty dy\, y^{-\sigma-1}\,\left[y\,(I_n\,K_n)'(y)\right]^{2p}, \qquad \text{for } -1<\mathrm{Re}\,\sigma<0.
\end{aligned} \tag{36}$$

Note that we are commuting a ξ-expansion with a process of analytic extension which sidesteps σ-poles (i.e., s-poles). Yet, since the ξ-dependence has no problematic traits, this should be correct, and we write

$$\begin{aligned}
\zeta_{\Lambda(D=2)}(\sigma) &= \sum_{n=0}^{\infty} d_n\,\zeta_n(\sigma) \;= \sum_{p\geq 1}\zeta_{\Lambda(D=2)}^{(2p)}(\sigma)\,\xi^{2p},\\
\zeta_{\Lambda(D=2)}^{(2p)}(\sigma) &= -\frac{1}{p}\,a^{\sigma}\,\frac{\sigma}{\pi}\sin\left(\frac{\pi\sigma}{2}\right)\sum_{n=0}^{\infty} d_n\,A_n^{(2p)}(\sigma).
\end{aligned} \tag{37}$$

If we just want to keep the terms $\sim\xi^2$ in $\mathcal{E}_C$, it will be enough to maintain the $p=1$ contribution, which can be rewritten in the way

$$\zeta_{\Lambda(D=2)}^{(2)}(\sigma) = -a^{\sigma}\,\frac{\sigma}{\pi}\sin\left(\frac{\pi\sigma}{2}\right)\int_0^\infty dy\, y^{-\sigma-1}\,F(y), \qquad F(y)\equiv\sum_{n=-\infty}^{\infty}\left[y(I_nK_n)'(y)\right]^2, \tag{38}$$

where we have also taken into account (28) and the fact that $\zeta_{-n}(\sigma) = \zeta_n(\sigma)$. An integral representation of the $F(y)$ is already available in eq.(14). From there, we proceed as in the derivation of eq.(15), i.e., we do the variable change $u \equiv |\sin(\varphi/2)|$ and find

$$F(y) = \frac{8\,y^2}{\pi} \int_0^1 \frac{du}{\sqrt{1-u^2}}\, u^2\, K_1^2(2uy). \tag{39}$$

With this, we go back to eq. (38) and focus on the integral

$$\mathcal{F}(\sigma) \equiv \int_0^\infty dy\, y^{-\sigma-1}\, F(y) = \frac{8}{\pi} \int_0^1 \frac{du}{\sqrt{1-u^2}}\, u^2 \int_0^\infty dy\, y^{-\sigma+1}\, K_1^2(2uy). \tag{40}$$

The y-integration is evaluated with the help of formula 6.576.4 in ref.[11]. Then, the remaining u-integral is immediate using forumla 3.251.1 in the same book. As a result,

$$\mathcal{F}(\sigma) = \frac{1}{2\sqrt{\pi}}\, \frac{\Gamma\left(\frac{4-\sigma}{2}\right)\Gamma^2\left(\frac{2-\sigma}{2}\right)\Gamma\left(-\frac{\sigma}{2}\right)}{\Gamma(2-\sigma)\Gamma\left(\frac{\sigma+2}{2}\right)}, \tag{41}$$

which has a zero of order one at $\sigma = -2$ by virtue of the singularity of $\Gamma\left(\frac{\sigma+2}{2}\right)$. Putting it into eq.(38) and expanding near $\sigma = -2$, we find

$$\zeta^{(2)}_{\Lambda(D=2)}(\sigma) = -a^\sigma\, \frac{\sigma}{\pi} \sin\left(\frac{\pi\sigma}{2}\right) \mathcal{F}(\sigma) = \frac{1}{a^2}\left[\frac{1}{6}(\sigma+2)^2 + \mathcal{O}\left((\sigma+2)^3\right)\right], \tag{42}$$

which provides the desired analytic extension to Re $\sigma = -2$. The crucial point is that it has a zero of order two at $\sigma = -2$ and, therefore, $\zeta^{(2)}_{\Lambda(D=2)}(-2) = 0$ and $\zeta^{(2)\,\prime}_{\Lambda(D=2)}(-2) = 0$. This, together with eqs. (25) and (30), leads to $\mathcal{E}_C = 0 + \mathcal{O}\left(\xi^4\right)$, i.e.,

$$\mathcal{E}_C^{(2)} = 0, \tag{43}$$

which was numerically found in ref.[7] [4] (see also [8, 9]).

3.2 Higher-order corrections in ξ^2

In order to know new corrections in ξ^2, one has to keep the next ξ^2-terms in eqs. (36), (37). However, unlike $A_n^{(2)}(\sigma)$, the $A_n^{(2p)}(\sigma)$ integrals with $p \geq 2$ are already finite at $\sigma = -2$, because $[y(I_nK_n)'(y)]^{2p} \sim \dfrac{1}{(2y)^{2p}}$ as $y \to \infty$. Thus, the restriction to Re $\sigma > -1$ in (36) is

[4] Incidentally, the comments made by one the authors of the present letter led to the correct numerical figure in ref.[7].

caused only by the presence of $p = 1$, while, for $p \geq 2$, it suffices to numerically evaluate all the necessary $A_n^{(2p)}(\sigma)$'s at $\sigma = -2$, i.e.,

$$\zeta_{\Lambda(D=2)}^{(2p)}{}'(-2) = -\frac{1}{p\,a^2}\sum_{n=0}^{\infty} d_n\, A_n^{(2p)}(-2). \tag{44}$$

A posteriori, we have verified that each term decreases quickly enough with n and that the n-summation has a numerically acceptable behaviour. Then, the finiteness of each of these sums confirms that $\zeta_{\Lambda(D=2)}^{(2p)}(-2) = 0$, and the ξ^{2p}-contributions to $\mathcal{E}_C$ are

$$\begin{aligned} \mathcal{E}_C^{(2p)}\,\xi^{2p} &= \hbar c\,\frac{1}{2}\,\zeta_{\Omega(D=3)}^{(2p)}{}'(-1)\,\xi^{2p} = \hbar c\,\frac{1}{4\pi}\,\zeta_{\Lambda(D=2)}^{(2p)}{}'(-2)\,\xi^{2p} \\ &= -\hbar c\,\frac{\xi^{2p}}{4\pi\,p\,a^2}\sum_{n=0}^{\infty} d_n\, A_n^{(2p)}(-2), \qquad\qquad \text{for } p \geq 2, \end{aligned} \tag{45}$$

where the meaning of $\zeta_{\Omega(D=3)}^{(2p)}(s)$ is obvious. When $p = 2$, including all the n-values up to $n_{\max} \sim 120$, we have found $\sum_{n\geq 0} d_n\, A_n^{(4)}(-2) \simeq 0.19108$. This and formula (45) yield

$$\mathcal{E}_C^{(4)}\,\xi^4 = -0.0076028\,\frac{\hbar c}{a^2}\,\xi^4, \tag{46}$$

in agreement with ref. [9]. As remarked there, the negative sign means that the involved Casimir forces are attractive. Physical implications concerning the flux tube model for confinement have been discussed in that work.

In fact, the higher p, the fewer terms are needed in the n-series for obtaining reliable figures. For $p > 2$ we have found many contributions, but we just list the first ones:

p	3	4	5	6	7	...
$\mathcal{E}_C^{(2p)} a^2/(\hbar c)$	-0.0022637	-0.0010807	-0.0006202	-0.0003972	-0.0002737	...

As argued in refs.[7] or [8], the special value $\xi^2 = 1$ should reproduce the perfectly-conducting case $\mathcal{E}_{\mathrm{C\ (p.c.)}}\, a^2/(\hbar c) = -0.01356\ldots$ [16, 10]. Taking all the contributions up to $p = 7$, we obtain $\mathcal{E}_C\, a^2/(\hbar c) \simeq -0.01224$, with a 10% relative error. This is not too surprising, as the ξ^2-expansion comes from a logarithimc series, and a slow numerical convergence at $\xi^2 = 1$ is expectable. Including all the terms up to $p = 200$ we have found $\mathcal{E}_C\, a^2/(\hbar c) \simeq -0.01354$, with a 0.15% relative error.

4 Conclusions

The ultimate consequences of any result about Casimir effect are not easy to foresee, as the domain of applicability of this concept has been expanding beyond what could be considered 'traditional' areas of field theory. For instance, we have recent examples of these ideas in spacetime evolution and quantum cosmology [17]. Proposals haven even been made about possible ways of extracting work from the vacuum energy [18].

In the present letter we have confirmed the expectation that the ξ^2-contribution to the Casimir energy for a dilute-dielectric cylinder, infinitely long, and under the condition of light-velocity conservation, would have to vanish. Numerically speaking, this had been noticed with very high accuracy in several articles, starting with ref.[7], but in the present letter we have been able to derive it as an *exact* result (eqs.(24) and (43)). Another new aspect lies in applying the method developed in [4], i.e., the use of summation theorems for infinite series of Bessel functions, which has spared us the handling of Debye expansions (see also the application of this method in ref.[19] and [20], but this time in connection with the problem of refs.[21]).

Moreover, by a numerical evaluation, and within the complete zeta function regularization framework, we have reobtained the ξ^4-contribution calculated in ref.[9], (our eq. (46)), which is negative. This constitutes the first deviation from zero and shows that, although at a higher order, the Casimir energy of this system would tend to contract the cylinder. Even higher corrections in ξ^2 have also been found (table in sec. 3).

Our spectral zeta function has been constructed like in ref.[10]. Other variants of the zeta function procedure, which differ from ours at some particular steps, are also in circulation (e.g. ref.[15] or [8]). We regard them as slightly different formulations of one common underlying principle. In particular, ref.[15] illustrates the advantages of dealing with the total zeta function as a whole object, rather than a series of partial-wave zeta functions.

A Appendix: The Meijer G function

Here we state some facts about the Meijer G function, which is defined by the integral

$$G_{pq}^{mn}(a_{\rm list}, b_{\rm list}, x) = \frac{1}{2\pi i}\int_L ds\, \frac{\prod_{j=1}^m \Gamma(b_j - s)\prod_{j=1}^n \Gamma(1-a_j+s)}{\prod_{j=m+1}^q \Gamma(1-b_j+s)\prod_{j=n+1}^p \Gamma(a_j - s)} x^s. \qquad (47)$$

The different integration paths L can be found, for example, in [12], as most of the other properties we use. By simple variable changes one may prove numerous identities such as:

$$x^n G_{pq}^{mn}(a_{\rm list}, b_{\rm list}, x) = G_{pq}^{mn}(a_{\rm list}+n, b_{\rm list}+n, x) \qquad (48)$$

and

$$G^{mn}_{pq}(a_{\rm list}, b_{\rm list}, \frac{1}{x}) = G^{mn}_{qp}(1 - b_{\rm list}, 1 - a_{\rm list}, x) \tag{49}$$

which we have used in secs. 2.1, 2.2.

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